THE USE OF TRIGONOMETRIC FUNCTIONS IN AGRONOMIC ANALYSIS

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ABSTRACT

The advantages of mathematical curve fitting to series of data points are noted and the more commonly fitted curves reviewed. For data points of processes which are of a cyclic nature, the fitting of an equation made up of sine and cosine functions is proposed. One method is by fitting the equation by linear regression analysis. The equation can be transformed to contain only a cosine function, and the parameters of the equation represent meaningful values. If the cycle appears to be asymmetric, higher order sine and cosine terms can be included. Another method is by Fourier analysis, which is computationally simpler, but the data points must be equidistant along the length of the cycle. Examples given show the fitting of cyclic curves to rainfall data; to experimental results of a month-by-month sucrose % cane investigation; and to monthly mill sucrose % cane data. The last-mentioned equation obtained is used for calculating the effects of timing of the milling season on seasonal average sucrose % cane.

Introduction

Often the results of an investigation consist of a series of paired points, and are usually displayed in graphical form with the values of the independent variable on the horizontal and the dependent variable on the vertical axis, to illustrate the functional relationship between them. Due to errors in measurement, random interference of unknown factors, etc., these points will generally not form a smooth curve when connected up in sequence by a series of straight lines. It is therefore necessary to estimate and draw a smooth curve through these points to obtain the required relationship. One method is simply to draw the curve by eye and free hand, possibly aided by a straight-edge or French curves. Another method, which is becoming more of a practical proposition in these days of electronic computers to perform the drudgery of calculation, is to fit appropriate mathematical equations to the points. These equations can then be plotted on the graphs, and will represent the estimated relationship between the variables.

Advantages of fitting mathematical equations instead of drawing free-hand curves

(1) It avoids human bias and inconsistency. No two persons will draw the same free-hand curve through a series of points on the graph, but for a given form of mathematical equation there can only be one curve to fit a given set of data.

(2) A theoretical investigation of the problem may have led to a certain form of equation, and if the object of the experiment was to confirm this theory, the procedure will be to fit that equation to the points. Numeric measures of how good the fit is can be determined, which will in turn determine the validity of the theory or hypothesis.

(3) The values of the constants obtained in the fitted equation (known as parameters) can often provide useful and meaningful information.

(4) Having the relationship available in the form of a mathematical equation makes it possible to subject it to further mathematical manipulations, such as differentiation or integration or incorporating it in a larger mathematical model.

(5) When there is more than one independent variable, it is no longer possible to represent the situation on a two-dimensional paper surface, except in the form of a contour map, where the curves to be determined will be the contours. Mathematical curve fitting procedures can take in more than one independent variable.

Linear regression analysis

Any relationship which can be reduced to the form: $y = A_0 + A_1x_1 + A_2x_2 + \ldots\ldots + A_nx_n$ is known as a linear relationship between the dependent variable y and the independent variables $x_1, x_2, \ldots\ldots, x_n$. The method of fitting the equation is known as multiple linear regression analysis ("multiple" when there is more than 1 independent variable), and is described in many textbooks.[4]

The calculating procedure is tedious, and all computer manufacturers have programs available for performing this operation.

The coefficients A_0, A_1, A_2, etc., are the parameters of the equation, and the whole process of fitting the equation revolves around determining that set of values of the parameters which will result in the lowest sum-of-squared deviations of the data points from the fitted line.

The linear fit is widely used because the computational procedure is straight-forward and precise, i.e. there is no need to do the fit through a series of successive approximations which, even in an electronic computer, can be time-consuming. It therefore always is desirable to try fitting curves which are reducible to the linear form.

Examples of equations for curve fitting

The investigator should already have some idea what form the smooth curve through the points should have, and choose the appropriate type of equation.

(1) *Straight line*

In its simplest form, the linear relationship reduces to the form:

$$y = A + Bx,$$

and has been described by Christianson[3] in an earlier SASTA publication. This fit will be used when the points appear to lie on a straight line or if there are theoretical reasons for the relationship to be a straight line.

(2) *Quadratic curve*

This takes the form of:

$$y = A + Bx + Cx^2$$

and is also known as a parabola.

This might not look like a linear relationship because of the presence of the x^2 term, but if we consider x as one independent variable and x^2 as another, the equation is linear and lends itself to multiple linear regression analysis.

The parameters have the following significance:

(*a*) A = Intercept on the y-axis.

(*b*) The value of C determines the sharpness of curvature: the larger the absolute value of C, the sharper the curvature.

(*c*) If C is positive, the open end of the parabola is upwards, and vice-versa.

(*d*) The position of the minimum (or maximum) is at a value of independent variable $= -\dfrac{B}{2C}$.

Fig. 4 in Appendix A illustrates the shape of a couple of parabolas.

A parabolic fit will often be used when there is a curved rather than a straight line relationship between the variables, and the curvature is fairly gentle and consistently in one direction.

(3) *Exponential relationship*

This is a relationship in the form of:

$$y = A \cdot B^t.$$

In these applications the independent variable usually is time, and the symbol t has been chosen instead of x.

Upon taking logarithms, we obtain:

$$\log y = \log A + t \log B.$$

If we now consider log A and log B as the parameters or constants of the equation, t as the independent and log y as the dependent variables, the transformed equation is linear.

This type of fit is especially applicable when dealing with growth of a compound interest type, e.g. population growth. The term B, which must always be positive, represents the factor of increase in y per unit increase of the independent variable t, and will be greater than 1 (i.e. log B will be positive) if the population increases with time, and less than 1 (log B negative) if the population decreases. Fig. 5 in Appendix A illustrates both cases.

Cyclic processes

Many processes which depend on climate and extend over a reasonably long period of time, say 1 year or longer, will be influenced by the seasonal effect, i.e. they will reach a maximum at a certain time of the year and a minimum at another time (often about 6 months distant). The average length of the cycle will of course be 1 year. The trigonometric functions sin x and cos x both show this characteristic of periodicity, and fluctuate between the limits of + 1 and −1, with a period length of 360°, or 2π radians, as illustrated in Fig. 6 of Appendix A. It is therefore logical to use these functions for building up a fitted equation to any data which exhibits periodicity.

Fitting trigonometric functions by multiple linear regression analysis

Here we let the fitted equation take the form:

$$y = A + B \cos(30t) + C \sin(30t),$$

where t is a numerical representation of the calendar months of the year, e.g. t = 1 for January, t = 2 for February, up to t = 12 for December. The factor 30 is used when the angle of the trigonometric functions is to be expressed in degrees, so that for 12 months of the year we have a full circle or cycle of $360° = 12 \times 30°$. If it were more suitable, t could have been expressed as say the week number, ranging from 1 to 52, and the factor would be $\dfrac{360}{52} = 6{,}92$.

If the angles were to be expressed in radians, the factor would be $2\pi/12 = \pi/6$ for t in terms of months. Although it appears more clumsy, radians have to be used instead of degrees when performing any operations of differentiation or integration on these functions, and most computers require that the angles of cosine and sine functions should be expressed in radians and not degrees.

This function again is not linear in t, but is linear if we consider cos (30t) and sin (30t) as two separate variables.

This function can be transformed into a more meaningful form, as follows:

It can be shown that[2] in general,

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

We can multiply and divide our fitted equation by $\sqrt{B^2 + C^2}$ as follows:

$$y = A + \sqrt{B^2 + C^2} \times \left[\frac{B}{\sqrt{B^2 + C^2}} \cos(30t) + \frac{C}{\sqrt{B^2 + C^2}} \sin(30t)\right]$$

Putting $D = \sqrt{B^2 + C^2}$, and defining angle ø as:

$$\cos ø = \frac{B}{\sqrt{B^2 + C^2}}, \quad \sin ø = \frac{C}{\sqrt{B^2 + C^2}},$$

the equation becomes:

$$y = A + D\,[\cos(30t)\cos ø + \sin(30t)\sin ø],$$

which simplifies to

$$y = D\cos(30t - ø).$$

Defining p as ø = 30p, we obtain

$$y = A + D\cos[30(t - p)], \text{ where}$$

A = neutral line about which the values oscillate.

D = amplitude of the oscillations.

p = time at which the function reaches its peak. The function cos x always reaches its maximum of + 1 at a value of x = 0. In this case, it will happen when t = p, so that 30(t − p) = 0.

The maximum value of the function will be A + D, and the minimum A − D.

Asymmetric cyclic curve

The cosine curve in the foregoing discussion is symmetrical, but would not provide a good fit for a cyclic curve which shows a pronounced deviation from symmetry, e.g. by rising faster than what it subsequently falls, or dwelling longer in the region of the maximum than the minimum.

The fit can be improved by including higher-order trigonometric functions in the equation, e.g. by adding the equivalent of cos 2x and sin 2x:

$$y = A + B_1\cos(\pi t/6) + C_1\sin(\pi t/6) + B_2\cos(\pi t/3) + C_2\sin(\pi t/3).$$

The more terms of successively higher orders which are included, the better will be the fit, but the danger of "over-fitting" would increase, meaning that the curve will attempt to go through all points, including outliers.

The second order terms in the above equation can also be reduced to a single cosine term of the form:

$$D_2\cos\left[\frac{\pi}{3}(t - p_2)\right],$$

but it is hard to form a physical concept of the parameters A, D_1, D_2, p_1 and p_2 in the resultant equation.

The use of Fourier analysis

Provided that the points for which data is available are spread at equal distances along the axis of the entire cycle, we can make use of Fourier analysis to fit a combination of trigonometric functions, which not only is computationally simpler, but also can take account of asymmetry of the cycle.

It can be shown that any periodic function F(t) with period 2π radians which does not have discontinuities or "kinks" can be expressed as an infinite series of the form:[8]

$$F(t) = \frac{a_0}{2} + a_1\cos t + b_1\sin t + a_2\cos 2t + b_2\sin 2t + \ldots\ldots + a_r\cos rt + b_r\sin rt + \ldots\ldots$$

$$\text{where } a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos rt\,dt, \quad r = 0, 1, 2, \ldots\ldots$$

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin rt\,dt, \quad r = 1, 2, \ldots\ldots$$

When the estimates Z_t of the values of F(t) are available for only m specific equidistant values along the cycle of t = 1, 2, ..., m, the values of a_r and b_r can be estimated by:[1]

$$a_r = \frac{2}{m}\sum_{t=1}^{m} Z_t\cos(2\pi rt/m)$$

$$b_r = \frac{2}{m}\sum_{t=1}^{m} Z_t\sin(2\pi rt/m)$$

If monthly values are available, m = 12.

The values of the coefficients $a_0/2$, a_1, b_1, a_2 and b_2 will be exactly the same as the parameters A, B_1, C_1, B_2 and C_2 obtained by linear regression analysis in the equation:

$$y = A + B_1\cos(\pi t/6) + C_1\sin(\pi t/6) + B_2\cos(\pi t/3 + C_2\sin(\pi t/3),$$

and the calculating procedure is far simpler.

Unfortunately, the Fourier analysis technique cannot be used for say the mill sucrose % cane values, because of the gap in data over the off-crop months.

Example 1: Rainfall data

Fig. 1 shows the average monthly rainfall figures, as recorded at Mount Edgecombe Experiment Station for the past 47 years. As to be expected, they exhibit periodicity, but it is fairly apparent that the cycle is not symmetrical. In particular, there seem to be more relatively high rainfall months during the season than relatively low rainfall months.

On the graph, a first-order trigonometric function (i.e. symmetric) has been plotted, as well as a second-order trigonometric function.

It is obvious, even by visual observation, that the second-order equation provides a better fit to the data.

Example 2: Sucrose yield experiment

Average sucrose % cane values obtained by Gosnell and Koenig[5] for NCo 376 cane over an 18-month period in Experiment 1 at the RSA Experiment Station

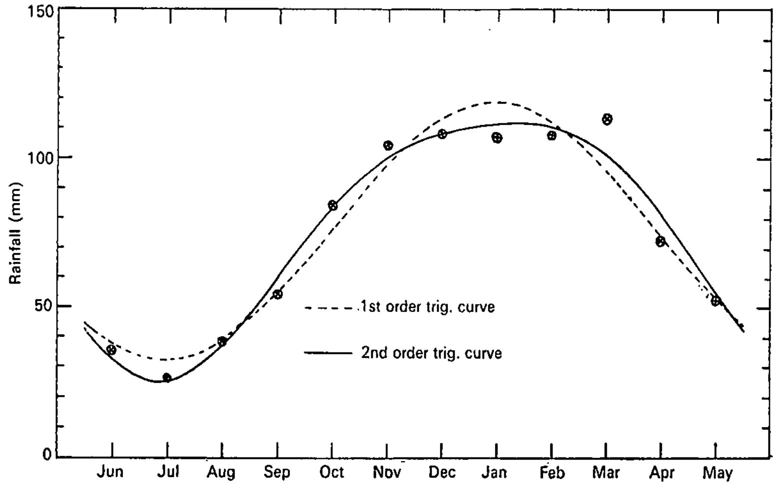

FIGURE 1 Average monthly rainfall at Mount Edgecombe.

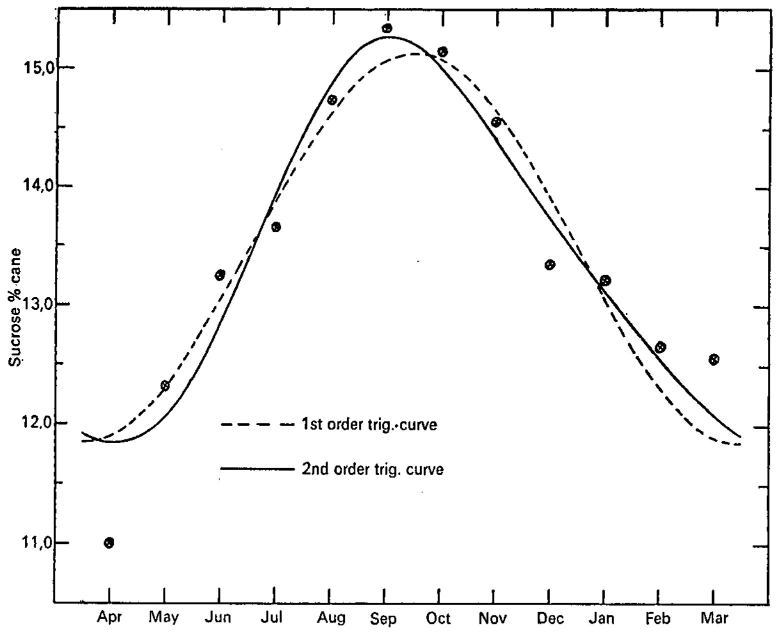

FIGURE 2 Monthly sucrose % cane for NCo 376, R.S.A. Experiment Station.

are repeated in Fig. 2. Here we again find that the values are not forming a symmetric cycle, as the hump of the curve appears to be slightly skewed towards the left. A first-order and a second-order curve have again been fitted to the data.

Example 3: Monthly mill sucrose % cane

First-order trigonometric equations were fitted to the monthly sucrose % cane figures of each of the 5 Hulett mills: Mount Edgecombe, Darnall, Amatikulu, Felixton and Empangeni, for the years 1962/63 to 1972/73. The seasons 1965/66, 1968/69 and 1970/71 were left out because they were abnormal, in that the mills ran short of cane and had to stop fairly early in the season. By way of comparison, quadratic equations were also fitted to the data, as per Santamaria *et al*,[7] but no weighting toward the more recent years was used. On the time scale, 1st May = 5,00, etc., but when moving on into the next calendar year of the same season, 1st January = 13,00, 1st February = 14,00, etc. Because the cycle time of the period is 12 time units ($12 \times \pi/6 = 2\pi$), the values of trigonometric term will not be affected by this change.

The results are given in Table 1.

As regards the values of the multiple correlation coefficients, there seems little to choose between the goodness of fit for the two alternative curves over the range of values for which data was available.

If we, however, turn to Fig. 3, in which all the data for the Darnall mill as an example, is plotted, together with the two alternative fitted curves, it can be seen that there is a strong divergence between the two curves when they are extrapolated into the off-crop part of the season. On moving through the period of the off-crop from February to May, the quadratic curve shows a continually steeper drop in sucrose, ending in a spike before rising for the following season. The cosine curve on the other hand levels out during the off-crop and then rises again, which of course is

TABLE 1

Fitting curves to sucrose % cane milling results 1962/63 to 1972/73, excluding 65/66, 68/69, 70/71

First-order trig: $S \% C = A + B \cos (\frac{\pi}{6} t) + C \sin (\frac{\pi}{6} t)$

Quadratic: $S \% C = A + B \cdot t + C \cdot t^2$

Second-order trig: $S \% C = A + B_1 \cos (\frac{\pi}{6} t) + C_1 \sin (\frac{\pi}{6} t)$

$$+ B_2 \cos (\frac{\pi}{3} t) + B_2 \sin (\frac{\pi}{3} t)$$

		Multi. Corr.	A	B	C	D = Amplitude	P = Time of Peak
ME:	First-order trig	0,707	12,652	0,241 7	– 0,980 9	1,010 2	9,46
	Quadratic	0,698	5,852	1,615 4	– 0,084 86		9,52
	Second-order trig	0,708					
DL:	First-order trig	0,816	13,147	0,344 9	– 1,135 2	1,186 4	9,56
	Quadratic	0,816	5,341	1,833 4	– 0,095 13		9,64
	Second-order trig	0,818					
AK:	First-order trig	0,847	13,009	0,320 2	– 1,211 2	1,252 8	9,49
	Quadratic	0,847	4,892	1,921 6	– 0,100 34		9,58
	Second-order trig	0,849					
FX:	First-order trig	0,826	12,545	0,637 5	– 1,046 0	1,225 0	10,05
	Quadratic	0,832	3,300	2,070 0	– 0,103 64		9,99
	Second-order trig	0,830					
EM:	First-order trig	0,715	13,008	0,443 1	– 1,162 0	1,243 6	9,69
	Quadratic	0,715	4,644	1,947 3	– 0,100 39		9,70
	Second-order trig	0,715					
ALL:	First-order trig	0,746	12,875	0,396 6	– 1,101 6	1,170 8	9,66
	Quadratic	0,746	4,863	1,864 1	– 0,096 13		9,70

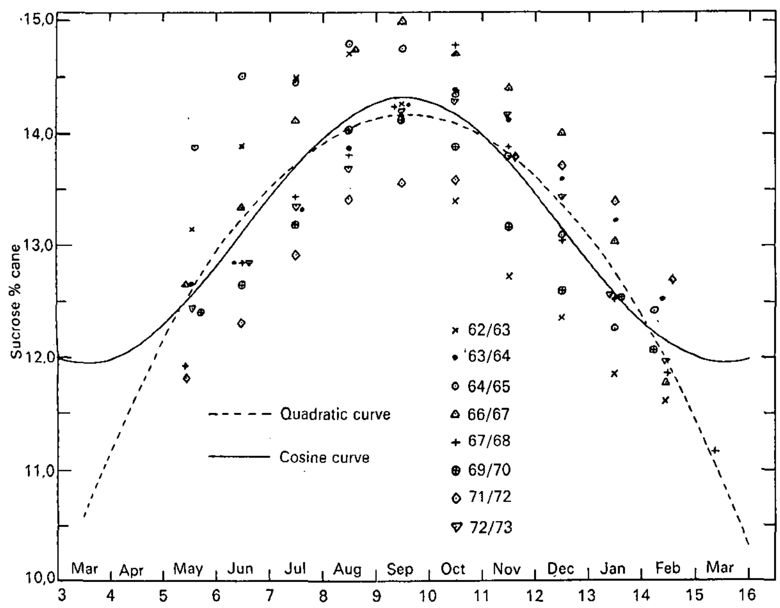

FIGURE 3 Monthly sucrose % cane values for Darnall mill.

TABLE 2

Fitting of first-order cosine curve to monthly sucrose % cane data for all Hulett mills, season-by-season and mid-season to mid-season

	Multiple Corr.	A	D	p
1962/63 Season	0,903	12,89	1,266 5	8,71
September 1962 — August 1963	0,907	12,68	1,076 4	3,12†
1963/64 Season	0,877	13,07	1,005 7	10,09
September 1963 — August 1964	0,840	13,57	0,988 5	2,76†
1964/65 Season	0,884	13,57	1,285 4	8,88
September 1964 — August 1965	0,760	13,24	1,079 3	3,56†
1965/66 Season*	0,799	12,66	0,817	7,52
September 1965 — August 1966	0,840	12,63	1,206 3	1,97†
1966/67 Season	0,888	13,25	1,272 8	9,44
September 1966 — August 1967	0,911	12,76	1,494 2	4,31†
1967/68 Season	0,882	12,56	1,479	10,06
September 1967 — August 1968	0,831	12,88	1,293 4	3,35†
1968/69 Season*	0,867	12,67	0,858 1	7,99
September 1968 — August 1969	0,768	12,21	1,009 3	3,29†
1969/70 Season	0,724	12,48	0,999 2	9,66
September 1969 — August 1970	0,862	13,07	0,951 6	2,03†
1970/71 Season*	0,902	12,60	1,768 0	8,15
September 1970 — August 1971	0,850	11,79	1,911 7	3,07†
1971/72 Season	0,829	12,60	0,968 3	10,60
September 1971 — August 1972	0,782	12,68	0,933 7	4,38†
1972/73 Season	0.926	12,63	1,465 6	10,05

* Considered to be abnormal seasons. † Time of minimum sucrose % cane value.

more like what has been observed in practice when sucrose % cane measurements were taken throughout the year, including the off-crop period, e.g. Gosnell.[5,6]

Another aspect to consider is whether the part-curve of sucrose % cane exhibits any asymmetry with regard to the amount of time spent in the high and low value regions. To test this, a regression analysis containing first- and second-order trigonometric functions was done, and the correlation coefficients are also included in Table 1. It is obvious that the inclusion of the second-order terms have hardly any effect on improving the fit, and in no case did the significance of the parameters B_2 and C_2 achieve an 80% level of confidence, thus implying that the sucrose % cane curve is symmetrical.

It was therefore decided that the first-order cosine curve would be the best equation to fit.

It is also interesting to fit the first-order cosine curve to the monthly sucrose % cane values for the Hulett mills ME, DL, AK, FX and EM combined, doing this season by season. The results are shown in Table 2.

Comparing with Table 1, note that the multiple correlation coefficients for fitting per season are higher than for fitting per mill. That, and the greater variation in values for amplitude D and time of peak p imply that there is more variation between seasons than between mills, at least within the geographical range over which the Huletts mills lie. The average time of peak is 9.20, with a standard deviation of 1.01, which includes the poor seasons.

In addition to the seasonal fits, the first-order cosine curve was also fitted over ranges of mid-season to mid-season for the combined mills data. The range was from September of one season, over the off-crop gap and up to August of the following season, and the results are included in Table 2. The goodness of fit is of the same order as the seasonal fits, and the amplitude D and time of minimum sucrose % cane (during off-crop) take on plausible values, providing further vindication of the first-order cosine curve fit.

Example 4: Calculation of effect of seasonal length on average mill sucrose % cane

An application of mathematical manipulation of a fitted equation is estimating the effect of seasonal length on the average mill sucrose % cane obtained during the season and thus the sucrose tonnage. This is done by integrating the sucrose % cane equation obtained in Example 3 between the time limits of start and end of season. This method will of course not be rigorously correct, because the timing of the season will automatically affect the timing of the ratoon crops, which could in turn affect the tons cane per hectare figures and hence the sucrose tonnage. Also, because the average age of crop harvested for the Hulett mills is around 18 months, the mill will run into seasonal cane approximately in the middle of the season, resulting in a drop in average age, and during the off-crop the average age will increase again. The longer the off-crop, the bigger will be the change in average age of cane harvested during the season, which will in itself affect the sucrose % cane values, quite apart from the effect of time of year. Unfortunately very little quantitative information is thus far available on the effects of ratooning month and cane age, and a very complicated calculation would have been required even if these relationships were known, so that these two aspects have to be ignored. For an average length of season, these effects were already influencing the historical sucrose % cane values upon which the fitted equations were based, so that ignoring these effects when altering the average length of season should not produce any significant error.

If, in general, we represent sucrose % cane by

$$S = A + D \cos \left[\frac{\pi}{6}(t - p)\right],$$

then the average sucrose % cane for a season which starts and ends at times t_1 and t_2 respectively, will be

$$\bar{S} = \frac{\int_{t_1}^{t_2} S\,dt}{t_2 - t_1}$$

$$= \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left[A + D \cos\left[\frac{\pi}{6}(t - p)\right]\right] dt$$

$$= \frac{1}{t_2 - t_1} \left[At + \frac{6D}{\pi} \sin\left[\frac{\pi}{6}(t - p)\right]\right]_{t = t_1}^{t = t_2}$$

$$= A + \frac{6D}{\pi(t_2 - t_1)} \left[\sin\left[\frac{\pi}{6}(t_2 - p)\right] - \sin\left[\frac{\pi}{6}(t_1 - p)\right]\right]$$

which gives average sucrose % cane as a function of t_1 and t_2.

Assuming that the season of length $\ell = t_2 - t_1$ months is to be symmetrically spaced about the peak, so that $t_2 - p = \ell/2$, $p - t_1 = \ell/2$, then

$$\bar{S} = A + \frac{12D}{\pi \ell}\left[\sin\left[\frac{\pi}{6} \cdot \frac{\ell}{2}\right]\right]$$

Taking again Darnall mill as an example, with

$$A = 13{,}15,$$

$$D = \sqrt{(0{,}344\,9)^2 + (-1{,}135\,2)^2}$$

$$= 1{,}186\,4,$$

the effect of the length of season on the average sucrose % cane can be calculated, if a 9-month season is considered as normal.

The results are shown in Table 3.

TABLE 3

Effect of length of season on average sucrose % cane for Darnall mill

Length of season (months)	Av. sucrose % cane	% Gain (+) or Loss (−) on 9-month season
5	14,025	3,8%
6	13,905	3,0%
7	13,775	2,0%
8	13,641	1,0%
9	13,506	0
10	13,377	− 1,0%
11	13,257	− 1,8%
12	13,150	− 2,6%

The effect of an asymmetric season (i.e. in which the mid-point does not coincide with the time of peak sucrose) on average sucrose can also be calculated.

If the shift from symmetry is represented by h months, keeping the total length of season constant at a value of L months,

$$\bar{S} = A + \frac{6D}{\pi L}\left[\sin\left[\frac{\pi}{6}\left(\frac{L}{2} + h\right)\right] + \sin\left[\frac{\pi}{6}\left(\frac{L}{2} - h\right)\right]\right]$$

In Table 4 the average sucrose % cane values for a 9-month season as a function of different values of shift from symmetry are shown.

TABLE 4

Effect of asymmetric 9-month season on average sucrose % cane for Darnall mill

Deviation from symmetry (months)	Av. sucrose % cane	% Gain (+) or Loss (−) on 9-month season
0	13,506	0
0,5	13,494	− 0,09%
1,0	13,458	− 0,36%
1,5	13,402	− 0,77%

Conclusions

In view of the fact that the sugar industry is strongly dependent on the weather cycle, there is much scope for the application of trigonometric functions to the analysis and mathematical modelling of all aspects of the sugar industry which are affected by the climatic cycle.

Changes in season length and shifts from symmetry make a surprisingly small difference to the mill average sucrose % cane.

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REFERENCES

1. Aitken, A. C. (1962). Statistical mathematics. p. 120-121. Oliver & Boyd (London), 8th edition.
2. Blakey, J. (1953). Intermediate pure mathematics. p. 132. Cleaver-Hume (London).
3. Christianson, W. O. (1960). Drawing a straight line through points on a graph. SASTA Proc. 34 : 67-69.
4. Davies, Owen L. (1954). Statistical methods in research and Production. p. 165-66. Oliver & Boyd (London), 2nd edition, revised.
5. Gosnell, J. M. and Koenig, M. J. P. (1972). Some effects of varieties on seasonal fluctuation in cane quality. SASTA Proc. 46 : 188-195.
6. Gosnell, J. M. (1967). The growth of sugarcane. Thesis for the degree of Doctor of Philosophy, Faculty of Agriculture, University of Natal. p. 81-82.
7. Santamaria, R., Aderman, C. and Saenz, A. (1973). Determination of the sucrose curve. Sugar y Azucar 68 (8): August 1973, p. 32-33.
8. Widder, D. V. (1947). Advanced calculus p. 324-357. Prentice-Hall (New York).

APPENDIX A

Examples of functions which can be reduced to the linear form

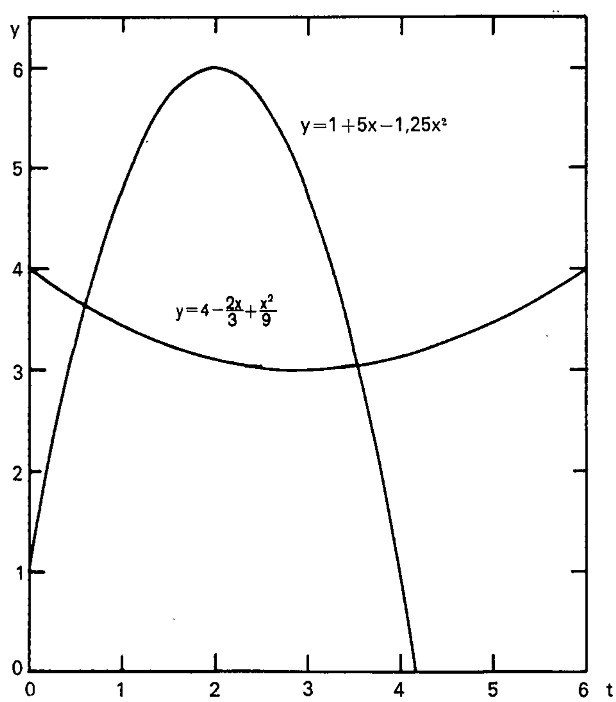

FIGURE 4 Illustration of quadratic curves.

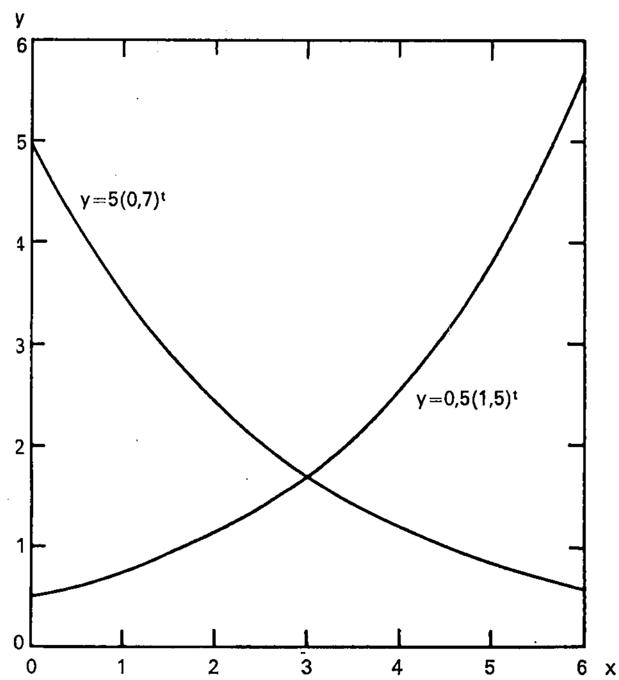

FIGURE 5 Illustration of exponential curves.

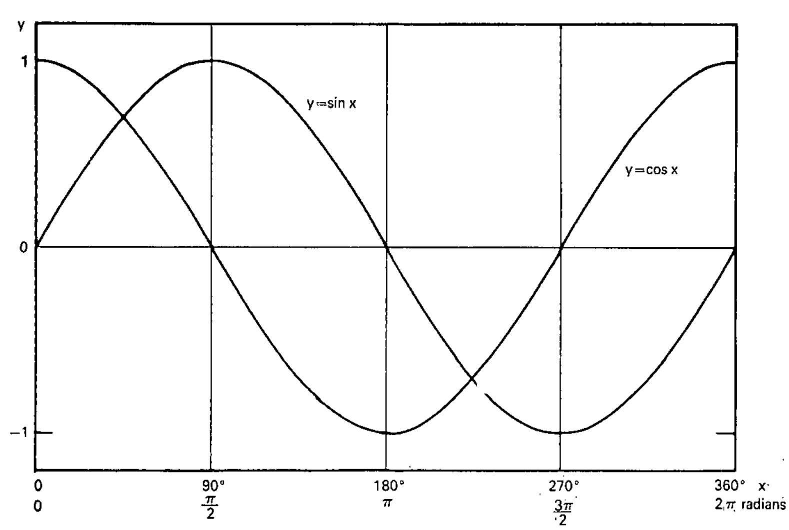

FIGURE 6 Illustration of trigonometric functions.